Operators That Attain their Minima

Xavier Carvajal[1], Wladimir Neves[1]

Abstract

In this paper we study the theory of operators on complex Hilbert spaces, which attain their minima in the unit sphere. We prove some important results concerning the characterization of the $\mathcal{N}^*$, and also $\mathcal{AN}^*$ operators, see respectively Definition 1.1 and Definition 1.3. The injective property plays an important role in these operators, and shall be established by these classes.

Contents

1 Introduction

We shall be concentrated on this paper in a class of bounded linear operators on complex Hilbert spaces, or on a subspace of it, which attains their minima on the unit sphere. Hereupon by a subspace, we are always saying a closed subspace. Certainly, the study of bounded linear operators that attain their minima have some similarities with the ones that achieve their norm as studied by the authors in [1]. Although, they not share the same characteristics, for instance the injectivity property plays an important role for that ones studied here, that is to say, the class of operators that attains their minima.

We are going to study mostly the operators that satisfy the $\mathcal{N}^*$ and $\mathcal{AN}^*$ properties, defined respectively in Definition 1.1 and Definition 1.3. The class

[1]Instituto de Matemática, Universidade Federal do Rio de Janeiro, C.P. 68530, Cidade Universitária 21945-970, Rio de Janeiro, Brazil. E-mail: *carvajal@im.ufrj.br*, *wladimir@im.ufrj.br*.
To appear in: Bulletin of the Brazilian Mathematical Society
Key words and phrases. Hilbert spaces, Bounded operators, Spectral representation.

of the $\mathcal{N}^*$ operators contains, for instance, the compact ones which are non-injective (see Proposition 1.2). Then, to introduce the theory, let H, J be complex Hilbert spaces and $\mathcal{L}(H,J)$ the Banach space of linear bounded operators from H to J. We emphasize the case that will appear most frequently later, namely $\mathcal{L}(H,H)=\mathcal{L}(H)$. Furthermore, we recall that, the space $\mathcal{L}(H,J)$ is a Banach space with the norm

$$\|T\| = \sup_{\|x\|_H \leq 1} \|Tx\|_J = \sup_{\|x\|_H = 1} \|Tx\|_J \tag{1.1}$$

and, it is well known that, if H has finite dimension, then the closed unit ball in H is compact (Heine-Borel Theorem) and the above *supremum* is a maximum. The important question whenever such a *supremum* is a maximum in the infinite dimensional case was studied by the authors in [1], where it is present many characterizations for operators that achieve their norm. Analogously, we now define the following value

$$[\,T\,] := \inf_{\|x\|_H = 1} \|Tx\|_J \tag{1.2}$$

and ask when such an $infimum$ is a minimum. This is one of the main issues of this article, which motivates the following

Definition 1.1. *An operator $T \in \mathcal{L}(H,J)$ is called to satisfy the property $\mathcal{N}^*$, when there exists an element x_0 in the unit sphere, such that*

$$[\,T\,] = \|T\,x_0\|_J.$$

We start the study by the following considerations:

1. An operator with zero minimum on the unit sphere should be non-injective in order to satisfy the property $\mathcal{N}^*$. Indeed, if there exists an element x_0 in the unit sphere, such that, $\|T\,x_0\|_J = [\,T\,] = 0$, it follows that $Tx_0 = 0$, and for T injective, $x_0 = 0$, which is a contradiction. Equivalently, if T is injective and satisfies the property $\mathcal{N}^*$, then $[T] > 0$.

2. If T is non-injective, then T attains its minimum and further $[T] = 0$. In fact, when T is non-injective, we have Ker $T \neq \{0\}$, and hence there exists an element $x \in$ Ker T, $x \neq 0$, such that $Tx = 0$. Therefore, $\|T(x/\|x\|)\| = 0 = [T]$.

3. Let us consider $T \in \mathcal{L}(H,J)$ with H finite dimensional. It is well-known that, $\dim T(H) \leq \dim H$, and since S is a compact set, it follows that $T(S)$ is compact. Therefore, applying the Weierstrass' Theorem, T attains its minimum on S. We have the following cases:

- If $\dim T(H) = \dim H$, then $\operatorname{Ker} T = \{0\}$ and thus T is injective. We conclude in this case that $[T] > 0$.

- If $\dim T(H) < \dim H$, then $\operatorname{Ker} T \neq \{0\}$ and T is non-injective. Thus $[T] = 0$.

Due the above considerations, we have the following complete characterization of the non-injective operators:

<table>
<tr><td colspan="2">$T \in \mathcal{L}(H,J)$, non-injective</td></tr>
<tr><td>$\dim H < \infty$</td><td rowspan="2">$T \in \mathcal{N}^*$,
$[\,T\,] = 0$</td></tr>
<tr><td>$\dim H = \infty$</td></tr>
</table>

Moreover, we have the following partial characterization of the injective operators:

<table>
<tr><td colspan="2">$T \in \mathcal{L}(H,J)$, injective</td></tr>
<tr><td>$\dim H < \infty$</td><td>$T \in \mathcal{N}^*$, $[\,T\,] > 0$</td></tr>
<tr><td rowspan="2">$\dim H = \infty$</td><td>$[\,T\,] = 0$, $T \notin \mathcal{N}^*$</td></tr>
<tr><td>$[\,T\,] > 0$, $T \in \mathcal{N}^*$?</td></tr>
</table>

On the other hand, if the dimension of H or the dimension of J are finite and $T \in \mathcal{L}(H,J)$, then there exists an x in the closed unit ball in H (indeed in the boundary, i.e. the unit sphere), such that

$$[\,T\,] = \|Tx\|_J.$$

Therefore, any operator of finite range satisfies the property $\mathcal{N}^*$. Moreover, an important class, which we have the complete characterization of the property $\mathcal{N}^*$, are the non-injective compact operators. Indeed, we have the following

Proposition 1.2. *Let $T \in \mathcal{L}(H,J)$ be a compact operator, with H infinite dimensional. Then, T satisfies the property $\mathcal{N}^*$ if, and only if, T is non-injective.*

Proof. 1. First, let us show that, any compact operator $T \in \mathcal{L}(H,J)$, with H infinite dimensional has $[T] = 0$. Indeed, let $\{e_n\}$ be an infinite orthonormal set in H. Therefore, applying Bessel's inequality, it follows for each $x \in H$

$$\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|_H^2.$$

Thus for each $x \in H$, we have $\lim_{n\to\infty} \langle x, e_n \rangle = 0$. Consequently, the sequence $\{e_n\}$ converges weakly to 0 in H. Now, since T is compact, $Te_n \to 0$ when $n \to \infty$. Thus

$$0 = \inf_n \|Te_n\|_J \geq \inf_{\|e\|=1} \|Te\|_J = [T].$$

2. Now, it follows from Consideration 2 in the preceding page that every non-injective operator on H attains its minimum on the unit sphere. Conversely let T be a compact operator on H that attains its minimum on the unit sphere. Since T is compact, from item 1 we have $[T] = 0$. Therefore, it follows from Consideration 1 in the preceding page that T is non-injective. $\square$

The restriction of a compact operator to a subspace is a compact operator. Although, we have seen for instance that, injectiveness is an important property w.r.t. the property $\mathcal{N}^*$. Since the restriction of a non-injective operator is not necessarily non-injective, it does not follow easy (even for the compact operator algebra) the following property.

Definition 1.3. *We say that $T \in \mathcal{L}(H, J)$ is an $\mathcal{AN}^*$ operator, or to satisfy the property $\mathcal{AN}^*$, when for all closed subspace $M \subset H$ ($M \neq \{0\}$), $T|_M$ satisfies the property $\mathcal{N}^*$.*

Remark 1.4. *Let $T \in L(H, J)$, if $\dim H < \infty$ or $\dim J < \infty$, then T satisfy the property $\mathcal{AN}^*$.*

We stress that by a subspace, we always mean a closed subspace, thus on the definition quoted above M is always closed. Moreover, it is not difficult to see that, one of the motivations to study the classes $\mathcal{N}^*$ and $\mathcal{AN}^*$ is related to show the injective property.

1.1 Notation and background

At this point we fix the functional notation used in this paper, and recall some well known results from functional analysis, we address the references [4], [6].

By $(H, \langle ., . \rangle)$ we always denote a complex Hilbert space, S will denote the unit sphere in H and B the closed unit ball in H

The space $\mathcal{L}(H)$ is not only a Banach space, but also an algebra. Moreover, we can define powers of $T \in \mathcal{L}(H)$, that is $T^0 = I$, where $Ix = x$ for all $x \in H$, and generally $T^n = TT^{n-1}$, $(n = 1, 2, \ldots)$. If $T \in \mathcal{L}(H, J)$, the adjoint operator of T is denoted by $T^* \in \mathcal{L}(J, H)$, which satisfies $\|T^*\| = \|T\|$.

An operator $P \in \mathcal{L}(H)$ is called positive, when $\langle P\,x, x\rangle \geq 0$, for all $x \in H$. Given an operator $T \in \mathcal{L}(H, J)$, we denote by P_T, the unique operator called the positive square root of T^*T, that is, $\langle P_T\,x, x\rangle \geq 0$ for all $x \in H$ and $P_T^2 = T^*T$. Moreover, for $T \in \mathcal{L}(H)$ we recall the polar decomposition of T, that is $T = UP_T$, where U is a unitary operator ($U^* = U^{-1}$). Not every $T \in \mathcal{L}(H)$ has a polar decomposition. See [6], Remark after Theorem 12.35.

As usual, if $x, y \in H$, then $x \perp y$ means that x is orthogonal to y, i.e. $\langle x, y\rangle = 0$. Additionally, if $M \subset H$, we define

$$M^\perp := \{x \in H : \langle x, y\rangle = 0, \text{for all } y \in M\},$$

that is the orthogonal complement of M, which is a (closed) subspace of H. If M is a subspace of H, hence closed by assumption, then we could write $H = M \oplus M^\perp$.

Let $T \in \mathcal{L}(H)$. The numerical range of T is defined as

$$W(T) := \{\langle Tx, x\rangle \in \mathbb{C};\, x \in S\}.$$

The Toeplitz-Hausdorff's Theorem asserts that $W(T)$ is a convex set. Now, if $T \in \mathcal{L}(H)$ is a self-adjoint operator, then $\|T\| = \sup_{x \in S} |\langle Tx, x\rangle|$. Therefore,

for $P \geq 0$, it follows that

$$\|P\| = \sup_{x \in S} \langle Px, x \rangle = \sup W(P). \tag{1.3}$$

Let $A \subset \mathbb{C}$ be a convex non-empty set. A number $\alpha \in A$ is said to be an extreme point of A, when $\alpha = t\,u + (1-t)\,v$, with $u, v \in A$ and $0 < t < 1$ implies, $\alpha = u = v$. Extreme points could be defined in more abstract set. Moreover, we recall the relation with convex sets given by the Krein-Milman theorem, see [6].

Finally, we recall some results and definitions in our paper [1].

Definition 1.5. *An operator $T \in \mathcal{L}(H, J)$ is said to satisfy the property $\mathcal{N}$, when there exists an element x in the unit sphere, such that*

$$\|T\| = \|T\,x\|_J.$$

Moreover, we say that T is an $\mathcal{AN}$ operator, or to satisfy the property $\mathcal{AN}$, when for all closed subspace $M \subset H$ ($M \neq \{0\}$), $T|_M$ satisfies the property $\mathcal{N}$.

Proposition 1.6. *Let $T \in \mathcal{L}(H)$ be a self-adjoint operator. Then, T satisfies $\mathcal{N}$ if, and only if $\|T\|$ or $-\|T\|$ is an eigenvalue of T.*

It follows from the above proposition that, if $P \in \mathcal{L}(H)$ is a positive operator and there exists an element $x_0 \in S$, such that $\|Px_0\| = \|P\|$, then

$$Px_0 = \|P\| x_0. \tag{1.4}$$

Likewise, since T satisfies $\mathcal{N}$ if, and only if P_T satisfies $\mathcal{N}$. Indeed,

$$\|T\| = \|P_T\| \quad \text{and} \quad \forall x \in H, \quad \|Tx\| = \|P_T x\|, \tag{1.5}$$

hence we have the following

Corollary 1.7. *An operator $T \in \mathcal{L}(H, J)$ satisfies $\mathcal{N}$ if, and only if $\|T\|$ is an eigenvalue of P_T.*

Now, we give the relation of the $\mathcal{N}$ condition and the adjoint operator.

Proposition 1.8. *Let $T \in \mathcal{L}(H, J)$, then T satisfies the condition $\mathcal{N}$ if, and only if the adjoint operator T^* satisfies $\mathcal{N}$.*

Lemma 1.9. *Let $T \in \mathcal{L}(H)$ be an self-adjoint operator. Then, T satisfies $\mathcal{N}$ if, and only if $\|T\|$ or $-\|T\|$ is an extreme point of the numerical range $W(T)$.*

2 The $\mathcal{N}^*$ operators

As we said through the introduction, the main issue of this paper is to study the operators that attain the minimum at the unit sphere. We begin showing some important characteristics of the $\mathcal{N}^*$ operators.

Lemma 2.1. *If T is self-adjoint operator on H, then for any $x \in H$ we have*

$$\|Tx\|^2 \geq [T] \langle Tx, x \rangle .$$

Proof. Consider the operator $S := T - [T]I$. Then, for each $x \in H$, we have

$$\|Sx\|^2 = \|Tx\|^2 - 2[T] \langle Tx, x \rangle + [T]^2 \|x\|^2 \geq 0.$$

It follows that, $\|Tx\|^2 + [T]^2 \|x\|^2 \geq 2[T] \langle Tx, x \rangle$. Now, since

$$\|Tx\|^2 \geq [T]^2 \|x\|^2,$$

the proof follows. □

Proposition 2.2. *Let $P \in \mathcal{L}(H)$ be a positive operator. Then,*

$$[P] = \inf \{ \langle Px, x \rangle ; x \in S \} .$$

Proof. Define $m := \inf \{ \langle Px, x \rangle ; x \in S \}$. If the kernel of P is different from zero, i.e. $\mathrm{Ker}P \neq \{0\}$, then $m = [P] = 0$ and the result is trivial. Now, suppose that $\mathrm{Ker}P = \{0\}$. We have

$$m = \inf \{ \langle Px, x \rangle ; x \in S \} \leq [P],$$

since $\langle Px, x \rangle \leq \|Px\| \|x\|$. On the other hand, using the positive square root of $P \geq 0$, it is known that, for all $x, y \in H$,

$$|\langle Px, y \rangle|^2 \leq \langle Px, x \rangle \langle Py, y \rangle . \tag{2.6}$$

Hence taking $y = Px$ in the above inequality, we have

$$\|Px\|^2 \leq \langle Px, x \rangle \left\langle P \left(\frac{Px}{\|Px\|} \right), \frac{Px}{\|Px\|} \right\rangle , \tag{2.7}$$

and combining with Lemma 2.1, we obtain

$$\forall x \in H, \quad [P] \leq \left\langle P \left(\frac{Px}{\|Px\|} \right), \frac{Px}{\|Px\|} \right\rangle .$$

Therefore, for all $z \in P(H)$

$$[P] \ \|z\|^2 \leq \langle Pz, z \rangle .$$

Consequently, as $\overline{P(H)} = H$, we conclude that

$$\forall z \in H, \quad [P] \ \|z\|^2 \leq \langle Pz, z \rangle ,$$

and it proves the proposition. □

Given $T \in \mathcal{L}(H)$, it is well known that, $\|T\| = \sup_{x,y\in S} |\langle Tx, y\rangle|$, see [6]. Therefore, we could conjecture that

$$[T] = \inf_{x,y\in S} |\langle Tx, y\rangle|.$$

In fact, this is false. Let us consider the following

Example 2.3. *Let $T : l^2 \to l^2$, $(x_j) \mapsto (\lambda_j x_j)$, with*

$$\lambda_1 > \lambda_2 > \cdots > \lambda > 0, \quad \lambda_j \searrow \lambda.$$

Then, $T \geq 0$ and it is easy to see that T does not satisfy $\mathcal{N}^*$. Indeed, we have $[T] = \lambda$, since

$$\|Tx\|^2 = \sum_{j=1}^{\infty} \lambda_j^2\, x_j^2 > \lambda^2\, \|x\|^2, \quad \forall x \neq 0, \|T\, \frac{x}{\|x\|}\| > \lambda$$

and, if (e_j) is the orthonormal canonical base of l^2, then $\|Te_j\| = \lambda_j \to \lambda$. On the other hand, $\inf_{x,y\in S} |\langle Tx, y\rangle| = 0$.

Remark 2.4. *If $P \geq 0$ and $[P] = \|P\|$, then $P = [P]I$. In fact by* (1.3) *and Proposition 2.2 we have for any $x \in H$ that*

$$[P]\, \|x\|^2 \leq \langle Px, x\rangle \leq \|P\|\, \|x\|^2 = [P]\, \|x\|^2,$$

and these inequalities give

$$\langle (P - [P]I)x, x\rangle = 0.$$

Therefore, since $P - [P]I \geq 0$, we conclude that $Px = [P]x$.

One observes that, if $P \geq 0$, then $P^n \geq 0$, $(n = 1, 2, \dots)$. This result is easily obtained by induction. Now, if $P \geq 0$, then it is not difficult to show using (2.6) that, for each $n \geq 1$

$$\|P^n\| = \|P\|^n. \tag{2.8}$$

The following proposition shows that $[\,.\,]$ also has the property (2.8).

Proposition 2.5. *Let $P \in \mathcal{L}(H)$ be a positive operator. Then,*

$$[P^n] = [P]^n.$$

Proof. By Proposition 2.2, we have

$$\begin{aligned} [P^2] &= \inf_{x\in S} \left\langle P^2 x, x\right\rangle = \inf_{x\in S} \langle Px, Px\rangle \\ &= [P]^2. \end{aligned} \tag{2.9}$$

Moreover, if $x \in S$, then

$$\begin{aligned} \left\langle P^{n+1}x, x \right\rangle =& \left\langle P^{n-1}(Px), Px \right\rangle \\ \geq& [P^{n-1}] \ \|Px\|^2 \\ \geq& [P^{n-1}] \ [P]^2. \end{aligned} \tag{2.10}$$

Consequently, we have that $[P^{n+1}] \geq [P^{n-1}] \ [P]^2$ and, it follows by induction that

$$[P^n] \geq [P]^n, \quad (n = 1, 2, \dots).$$

On the other hand, for any $x \in S$, $(n = 1, 2 \dots)$, by Lemma 2.1, we get

$$\begin{aligned} [P^{n-1}] \left\langle P^{n+1}x, x \right\rangle =& [P^{n-1}] \left\langle P^{n-1}(Px), Px \right\rangle \\ \leq& \left\| P^{n-1}(Px) \right\|^2, \end{aligned} \tag{2.11}$$

thus for any $n \in \mathbf{N}$, we obtain

$$[P^{n-1}] \ [P^{n+1}] \leq [P^n]^2.$$

By induction, we get

$$[P^n] \leq [P]^n, \quad (n = 1, 2 \dots).$$

Indeed, assuming that $[P^n] \leq [P]^n$, it follows that

$$[P^{n-1}] \, [P^{n+1}] \leq [P^n]^2 \leq [P]^{2n} = [P]^{n-1} \, [P]^{n+1} \leq [P^{n-1}] \, [P]^{n+1},$$

and therefore $[P^{n+1}] \leq [P]^{n+1}$, which proves the proposition. $\square$

Similarly to the property $\mathcal{N}$ see Proposition 1.6 and Lemma 1.9, we have the following

Proposition 2.6. *Let $P \in \mathcal{L}(H)$ be a positive operator.*

i) P satisfies $\mathcal{N}^*$ if, and only if $[P]$ is an eigenvalue of P.

ii) P satisfies $\mathcal{N}^*$ if, and only if $[P]$ is an extreme point of $W(P)$.

Proof. 1. In order to prove the item (i), first we suppose that P satisfies $\mathcal{N}^*$, i.e., there exists $x_0 \in S$, such that $\|Px_0\| = [P]$. Now, if $[P] = 0$, then it is obvious that 0 it is an eigenvalue. Therefore, we assume that $[P] > 0$ and, we have that

$$\left\langle (P^2 - [P]^2 I)x_0, x_0 \right\rangle = \|Px_0\|^2 - [P]^2 = 0.$$

Since $P^2 - [P]^2 I \geq 0$ and taking $z = Px_0 - [P]x_0$, it follows that

$$Pz + [P]z = 0.$$

Thus $\langle Pz, z \rangle = -[P] \ \|z\|^2$ and as $P \geq 0$ we concludes that $z = 0$, hence $[P]$ is an eigenvalue. Now, it is obvious that if $[P]$ is an eigenvalue of P, then P satisfies $\mathcal{N}^*$.

2. The proof of item (ii) is similar with that one given at Proposition 1.6. $\square$

The next example is an injective operator, which does not satisfy the $\mathcal{N}^*$ property.

Example 2.7. *Consider the operator of Example 2.3, that is*

$$T : l^2 \to l^2; \quad x \mapsto (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots),$$

with

$$\lambda_j \searrow \lambda > 0, \quad \lambda_1 > \lambda_2 > \dots .$$

It is not difficult to verify that, $T \geq 0$ is an injective operator. Moreover, we have that T does not satisfy $\mathcal{N}^*$ property, which follows also since the numerical range of T is the interval $(\lambda, \lambda_1]$ and $[T] = \lambda$ is not an extreme point of the numerical range.

Given an operator T on H which satisfies the $\mathcal{N}$ condition, it is not necessarily true that T^2 also satisfies $\mathcal{N}$. In fact, the following example shows an operator that satisfies $\mathcal{N}$ and, such that T^2 does not satisfy $\mathcal{N}$.

Example 2.8. *Let $T : l^2 \to l^2$, $(x_1, x_2, x_3, \dots) \mapsto (\lambda x_2, 0, \lambda_1 x_3, \lambda_2 x_4, \dots)$, with*

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda, \quad \lambda_j \nearrow \lambda.$$

Then, T satisfies $\mathcal{N}$ condition, since

$$\lambda = \|T\| = \|Te_2\|,$$

where $e_2 = (0, 1, 0, \dots)$. But, we could show easily that

$$T^2(x_1, x_2, x_3, \dots) = (0, 0, \lambda_1^2 x_3, \lambda_2^2 x_4, \dots),$$

does not satisfy the property $\mathcal{N}$.

Example 2.9. *Let $T : l^2 \to l^2$, $(x_1, x_2, x_3, \dots) \mapsto (\lambda x_2, 0, \lambda_1 x_3, \lambda_2 x_4, \dots)$, with*

$$\lambda_1 > \lambda_2 > \dots > \lambda > 0, \quad \lambda_j \searrow \lambda.$$

Then, T satisfies $\mathcal{N}^*$ condition, since

$$\lambda = [T] = \|Te_2\|,$$

where $e_2 = (0, 1, 0, \dots)$. But, similarly as above we could show that

$$T^2(x_1, x_2, x_3, \dots) = (0, 0, \lambda_1^2 x_3, \lambda_2^2 x_4, \dots),$$

does not satisfy the property $\mathcal{N}^*$.

Proposition 2.10. *Let $P \in \mathcal{L}(H)$, $P \geq 0$ and n be a positive integer.*

i) P satisfies $\mathcal{N}$ if, and only if P^n satisfies $\mathcal{N}$.

ii) P satisfies $\mathcal{N}^*$ if, and only if P^n satisfies $\mathcal{N}^*$.

Proof. 1. First, we show (i). When $P = 0$, the result is trivial, hence we assume $P > 0$. If P^n satisfies $\mathcal{N}$, then by Corollary 1.7, we have

$$P^n x_0 = \|P^n\| x_0 = \|P\|^n x_0,$$

for some $x_0 \in S$. Therefore, an algebraic manipulation gives

$$P\left(\frac{P^{n-1}x_0}{\|P\|^{n-1}}\right) = \|P\| x_0.$$

Now, since $\| P^{n-1}x_0/\|P\|^{n-1} \| \leq 1$, we obtain that P satisfies $\mathcal{N}$. In order to show that, if P satisfies $\mathcal{N}$ condition, then P^n satisfies $\mathcal{N}$, the proof follows easily applying Corollary 1.7.

2. The proof of the item (ii) is similar. $\square$

Proposition 2.11. *Let $P \in \mathcal{L}(H)$, $P \geq 0$, n and k be positive integers. Define*

$$T_n := \|P\|^n I - P^n, \quad \tilde{T}_n := P^n - [P]^n I.$$

i) P satisfies $\mathcal{N}^*$ <=> T_n satisfies $\mathcal{N}$ <=> $(T_n)^k$ satisfies $\mathcal{N}$.

ii) P satisfies $\mathcal{N}$ <=> $\tilde{T}_n^*$ satisfies $\mathcal{N}^*$ <=> $(\tilde{T}_n^*)^k$ satisfies $\mathcal{N}^*$.

Proof. 1. Let us show (i). If P satisfies $\mathcal{N}^*$, then by Proposition 2.6 there exists $x_0 \in S$, such that $Px_0 = [P]x_0$, it follows that $P^n x_0 = [P]^n x_0$, and since for each $x \in H$,

$$\langle T_n x, x\rangle = \|P\|^n \|x\|^2 - \langle P^n x, x\rangle \geq 0,$$

we have

$$\|T_n\| = \|P\|^n - [P]^n = \|P\|^n \|x_0\|^2 - \langle P^n x_0, x_0\rangle = \langle T_n x_0, x_0\rangle.$$

Consequently, we obtain that $T_n = \|P\|^n I - P^n$ satisfies $\mathcal{N}$. Now, if $T_n \geq 0$ satisfies $\mathcal{N}$, then there exist $x_0 \in S$, such that

$$\langle T_n x_0, x_0\rangle = \|P\|^n - \langle P^n x_0, x_0\rangle = \|T_n\| = \|P\|^n - [P]^n$$

and this implies that, $\langle P^n x_0, x_0\rangle = [P]^n = [P^n]$. By Proposition 2.10 we conclude that, P satisfies $\mathcal{N}^*$. Finally, since $T_n \geq 0$, by Proposition 2.10 T_n satisfies $\mathcal{N}$ if, and only if $(T_n)^k$ satisfies $\mathcal{N}$, which completes the proof of (i).

2. The proof of the item (ii) is similar. $\square$

Proposition 2.12. *Let $P \in \mathcal{L}(H)$, $P \geq 0$ and (p_n) be a sequence of polynomials with positive coefficients, such that*

$$p_n(P) \to S \quad \text{in } \mathcal{L}(H). \tag{2.12}$$

If P satisfies $\mathcal{N}$, then S satisfies $\mathcal{N}$.

Proof. If $\|Px_0\| = \|P\|$, for some $x_0 \in S$, then we have from Corollary 1.7 and Proposition 2.10 that

$$\begin{aligned} p_n(P)x_0 =& \alpha_0 x_0 + \alpha_1 P(x_0) + \cdots + \alpha_n P^n(x_0) \\ =& p_n(\|P\|)x_0. \end{aligned} \tag{2.13}$$

The equality (2.13) gives $\|p_n(P)\| = p_n(\|P\|)$ and by (2.12), we obtain

$$p_n(\|P\|) \to \|S\| .$$

Now, the convergence (2.12) and the equality (2.13) also imply that

$$p_n(\|P\|) \to \|Sx_0\| .$$

Therefore $\|S\| = \|Sx_0\|$. $\square$

Example 2.13. *Let $P \in \mathcal{L}(H)$ be a positive operator. If P satisfies $\mathcal{N}$, then the exponential operator* $\exp(P)$ *satisfies $\mathcal{N}$. Moreover, we have*

$$\|p_n(P)\| = \sum_{j=1}^{n} \frac{\|P\|^j}{j!} \to e^{\|P\|},$$

as $n \to \infty$. Consequently, we have $\| \exp(P)\| = e^{\|P\|}$.

3 The $\mathcal{AN}^*$ operators

In this section, we are going to study the operators that satisfy Definition 1.3, that is, the $\mathcal{AN}^*$ operators.

As already seen in [1], any compact operator T in $\mathcal{L}(H, J)$ is an $\mathcal{AN}$ operator. Indeed, if M is any closed subspace of H, then $T|_M$ is compact and therefore satisfies $\mathcal{N}$. Consequently, the algebra of compact operators carries $\mathcal{AN}$ out. Although, for the $\mathcal{AN}^*$ condition as we have showed at Example 2.7, with $\lambda = 0$, a compact operator T does not necessarily satisfy the $\mathcal{AN}^*$ property. Note that, if $T \in \mathcal{L}(H, J)$ is a compact operator with $[T] > 0$, which implies that $\dim H < \infty$ necessarily (see proof of Proposition 1.2), then T satisfies $\mathcal{AN}^*$.

Now, since an orthogonal projection is a partial isometry, it follows that any projection satisfies the properties $\mathcal{N}$ and $\mathcal{N}^*$. Although, it was showed in [1] that there exist a projection, which does not satisfy the $\mathcal{AN}$ property. Similarly, it is not necessarily true that each projection satisfies the $\mathcal{AN}^*$ property. In fact, we have the following

Example 3.1. *Let X be the subspace of l^2 of all x of the form*

$$x = (x_1, x_2, x_2, x_3, x_4, x_4, x_5, \dots)$$

and P is the projection on X, i.e., $P : l^2 \to l^2$,

$$P(x_1, x_2, x_3, \dots) = \Big(x_1, \frac{x_2 + x_3}{2}, \frac{x_2 + x_3}{2}, x_4, \frac{x_5 + x_6}{2}, \frac{x_5 + x_6}{2}, x_7, \dots\Big).$$

Now, let M be a subspace of l^2, defined as

$$M := \{x \in l^2 : x = (x_1, x_1, x_2, x_2, x_2, x_3, x_3, x_3, x_4, x_4, x_4, \dots)\}.$$

It follows that, $M \cap X = \{0\}$. Set $P|_M \equiv T : M \to l^2$, hence

$$T(x_1, x_1, x_2, x_2, x_2, \dots) = \Big(x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_2, \frac{x_2 + x_3}{2}, \frac{x_2 + x_3}{2}, x_3, \dots\Big).$$

For each $x \in M \cap S$, we compute the norm of Tx. First, we have

$$1 = \|x\|^2 = 2x_1^2 + 3\sum_{j=2}^{\infty} x_j^2, \tag{3.14}$$

hence it follows that

$$\begin{aligned}
\|Tx\|^2 &= \sum_{j=1}^{\infty} x_j^2 + 2\sum_{j=1}^{\infty} \big(\frac{x_j + x_{j+1}}{2}\big)^2 \\
&= x_1^2 + \sum_{j=2}^{\infty} x_j^2 + \frac{x_1^2}{2} + \sum_{j=2}^{\infty} x_j^2 + \sum_{j=1}^{\infty} x_j\, x_{j+1} \qquad (3.15) \\
&= \frac{2}{3} + \frac{x_1^2}{6} + \sum_{j=1}^{\infty} x_j\, x_{j+1},
\end{aligned}$$

where we have used (3.14). We take a convenient sequence $\{t^n\}$ contained in $M \cap S$, to show that T does not satisfy the $\mathcal{AN}^*$ condition. Indeed, we will show that, for all $x \in M \cap S$,

$$\|Tx\| > [T] = \frac{1}{\sqrt{3}}.$$

We consider the sequence $\{t^n\}_{n=1}^{\infty} \subset M \cap S$,

$$t^n = \{t_1^n, t_1^n, t_2^n, t_2^n, t_2^n, \dots t_n^n, t_n^n, t_n^n, \dots\}, \quad \|t^n\| = 1,$$

t^n defined by

$$t_j^n = \begin{cases} \dfrac{(-1)^j}{\sqrt{3(n-1)+2}} & j = 1, \dots, n, \\ 0 & j > n. \end{cases}$$

It follows that

$$\begin{aligned}\|Tt^n\|^2 &= \frac{2}{3} + \frac{(t_1^n)^2}{6} + \sum_{j=1}^{n-1} t_j^n\, t_{j+1}^n \\ &= \frac{2}{3} + \frac{1}{6(3(n-1)+2)} - \sum_{j=1}^{n-1} \frac{1}{3(n-1)+2} \\ &= \frac{2}{3} + \frac{7-6n}{6(3(n-1)+2)}.\end{aligned}$$

Then, we obtain

$$\lim_{n\to\infty} \|Tt^n\|^2 = \frac{2}{3} - \frac{6}{18} = \frac{1}{3}.$$

Now using (3.14)*, we have for any* $x \in M \cap S$

$$\begin{aligned}\|Tx\|^2 &= \sum_{j=1}^{\infty} x_j^2 + 2\sum_{j=1}^{\infty} \big(\frac{x_j + x_{j+1}}{2}\big)^2 \\ &= x_1^2 + \frac{1-2x_1^2}{3} + \sum_{j=1}^{\infty} \frac{\big(x_j + x_{j+1}\big)^2}{2} \\ &= \frac{1}{3} + \frac{x_1^2}{3} + \sum_{j=1}^{\infty} \frac{\big(x_j + x_{j+1}\big)^2}{2}.\end{aligned} \tag{3.16}$$

Consequently, for any $x \in M \cap S$ *we have* $\|Tx\|^2 \geq 1/3$ *and hence* $[T] = 1/\sqrt{3}$*. Therefore, we find out that* T *does not satisfy* $\mathcal{N}^*$*. Indeed, if there exists an element* $\tilde{x}$ *in* $S \cap M$*, such that* $\|T\tilde{x}\| = [T] = 1/\sqrt{3}$*, then by* (3.16)

$$\frac{\tilde{x}_1^2}{3} + \sum_{j=1}^{\infty} \frac{\big(\tilde{x}_j + \tilde{x}_{j+1}\big)^2}{2} = 0,$$

and this equation implies that $\tilde{x}_1 = 0$*. Moreover,* $\tilde{x}_j + \tilde{x}_{j+1} = 0$ *for* $j = 1, 2, \ldots,$ *it follows that* $\tilde{x}_j = 0$ *for* $j = 1, 2, \ldots,$ *which is a contradiction since that* $\|\tilde{x}\| = 1$*. Hence,* T *does not satisfy* $\mathcal{N}^*$*.*

Therefore, we have proved the following result.

Lemma 3.2. *Let* P *be an orthogonal projection. Then* P *does not necessarily satisfy* $\mathcal{AN}^*$ *property.*

The next proposition will be used as a proof of the next theorem, but it is important by itself.

Proposition 3.3. *Let* R *be an isometry on* H *and* $T \in \mathcal{L}(H)$ *an* $\mathcal{AN}^*$ *operator. Then,* TR *and* RT *satisfy the property* $\mathcal{AN}^*$*.*

Proof. Similar to that one given at Proposition 3.2 in [1]. ☐

Subsequently, we recall a well known definition for equivalent operators.

Definition 3.4. *The operators $T \in \mathcal{L}(H)$ and $S \in \mathcal{L}(J)$ are called unitarily equivalents, when there exists a unitary operator U on $\mathcal{L}(J, H)$, such that*

$$U^* T U = S.$$

In fact, if T and S are unitarily equivalents, then there is no criterion based only on the geometry of the Hilbert space, in such a way that, T could be distinguished from S. Therefore, since T and S are abstractly the same operator, it is natural to conjecture that some characteristic endowed by T must be satisfied by S, and vice versa.

Theorem 3.5. *Let T, S be two unitarily equivalent operators. Then, T is $\mathcal{AN}^*$ operator if, and only if S is an $\mathcal{AN}^*$ operator.*

Proof. Assume that U is a unitary operator such that $U^* T U = S$, hence $TU = US$. Since U is an isometry, by Proposition 3.3 if T satisfies $\mathcal{AN}^*$, then TU satisfies $\mathcal{AN}^*$. Moreover, it follows that, US also satisfies $\mathcal{AN}^*$. Once more, conforming to Proposition 3.3, we have that S satisfies property $\mathcal{AN}^*$. ☐

Remark 3.6. *Given $T \in \mathcal{L}(H, J)$, we recall that P_T was defined as the positive square root of T^*T. Therefore, T satisfies $\mathcal{AN}^*$ if, and only if P_T satisfies $\mathcal{AN}^*$, see* (1.5). *Consequently, it is enough to establish the condition $\mathcal{AN}^*$ for positive operators.*

Proposition 3.7. *An operator $T \in \mathcal{L}(H, J)$ satisfies the property $\mathcal{AN}^*$ if, and only if, for all orthogonal projection $Q \in \mathcal{L}(H)$, the composition TQ satisfies $\mathcal{N}^*$.*

Proof. Let M be a closed subspace of H and Q an orthogonal projection on M. Then, we have

$$[TQ] = [T|_M].$$

☐

Lemma 3.8. *Let $R \in \mathcal{L}(H)$ be an operator of finite rank. Then $I + R$ is an $\mathcal{AN}^*$ operator.*

Proof. We suppose that $\dim R(H) = n$. Hence we have

$$Rx = \sum_{j=1}^{n} \lambda_j \langle x, e_j \rangle \, e_j$$

where $\{e_j\}_{j=1}^n$ is an orthonormal set of H and $\lambda_j \geq 0$, $(j = 1, 2, \ldots n)$. Let M_n be the subspace generated by $\{e_1, \ldots, e_n\}$, thus we could write

$$H = M_n \oplus M_n^{\perp}.$$

Moreover, for any $x \in H$, $x = x_1 + x_2$, such that

$$x_1 = \sum_{j=1}^{n} \langle x, e_j \rangle \, e_j \quad \text{and} \quad x_2 = \sum_{\alpha \in A} \langle x, \tilde{e}_\alpha \rangle \, \tilde{e}_\alpha,$$

where $\{\tilde{e}_\alpha\}_{\alpha \in A}$ is an orthonormal basis of $M_n^\perp$, $\langle \tilde{e}_\alpha, e_j \rangle = 0$, for all $j = 1, \ldots, n$, $\alpha \in A$ and $\langle \tilde{e}_\alpha, \tilde{e}_\beta \rangle = \delta_{\alpha\beta}$ for each $\alpha, \beta \in A$. Now, define $T := I + R$, then for each $x \in H$,

$$Tx = \sum_{j=1}^{n} \langle x, e_j \rangle \, e_j + \sum_{\alpha \in A} \langle x, \tilde{e}_\alpha \rangle \, \tilde{e}_\alpha + \sum_{j=1}^{n} \lambda_j \langle x, e_j \rangle \, e_j.$$

Consequently, for each $x \in S$,

$$\|Tx\|^2 = 1 + \sum_{j=1}^{n} \left(\lambda_j^2 + 2\,\lambda_j \right) |\langle x, e_j \rangle|^2.$$

Therefore, if P is the finite range projection on M_n, then for any $x \in S$

$$\|TPx\| = \|Tx\|\,,$$

and as TP has finite range and therefore satisfies $\mathcal{AN}^*$, then T satisfies $\mathcal{AN}^*$. $\square$

Lemma 3.9. *Let H be a separable Hilbert space. If $P, Q \in \mathcal{L}(H)$ are two orthogonal projections such that, the dimension of their ranks and null spaces are infinite, then P and Q are unitarily equivalent.*

Proof. Since the rank and the null space of a projection are subspaces, there exist unitary operators $U_1 : P(H) \to Q(H)$ and $U_2 : \operatorname{Ker} P \to \operatorname{Ker} Q$. Now, we define $U : H \to H$, such that

$$U|_{P(H)} = U_1 \quad \text{and} \quad U|_{\operatorname{Ker} P} = U_2.$$

Hence it is clear that U as defined above is a unitary operator. Moreover, if $x \in H$, then $x = x_1 + x_2$ where $x_1 \in P(H)$ and $x_2 \in \operatorname{Ker} P$. From the definition of U_1 and U_2, we have

$$QUx = Ux_1 = UPx.$$

Therefore, P and Q are unitarily equivalents. $\square$

Theorem 3.10. *Let $Q \in \mathcal{L}(H)$ be an orthogonal projection. Then, Q satisfies the $\mathcal{AN}^*$ property if, and only if, the dimension of the null space or the dimension of the rank of Q is finite.*

Proof. If $\dim Q(H) < \infty$, then it is clear that Q satisfies $\mathcal{AN}^*$. Now, assume $\dim \operatorname{Ker} Q < \infty$. Then, we have $Q = I - P$, where P is a projection with finite rank. Therefore, by Lemma 3.8 Q satisfies the $\mathcal{AN}^*$ property.

Now, let us show that, if Q satisfies the $\mathcal{AN}^*$ property, then the dimension of the null space or the dimension of the rank of Q is finite, we show the contrapositive. Let $\dim Q(H)$ and $\dim \operatorname{Ker} Q$ be infinite. We consider two cases:

i) H separable. In this case, by Lemma 3.9, we have that Q is unitarily equivalent to the orthogonal projection of Example 3.1, which does no satisfy the $\mathcal{AN}^*$ condition. Consequently, by Theorem 3.5 Q does not satisfy $\mathcal{AN}^*$ either.

ii) H is not separable. In this point we will use the following

It is not difficult to prove that: If $P \in \mathcal{L}(H)$ is an orthogonal projection and J is a subspace of H, such that $P(H) \subset J$, then $P|_J \in \mathcal{L}(J)$ is also an orthogonal projection and moreover:

$$P(J) = P(H) \quad \text{and} \quad \operatorname{Ker} P|_J = J \cap \operatorname{Ker} P.$$

If $Q(H)$ is countable, we take $J \subset H$ be a separable Hilbert space, such that $Q(H) \subset J$ and $\dim(Q(H)^\perp \cap J) = \infty$. Thus by the claim above, we have that $Q|_J$ is an orthogonal projection, $Q|_J \in \mathcal{L}(J)$ satisfying

$$Q|_J(J) = Q(H) \quad \text{and} \quad \operatorname{Ker} Q|_J = J \cap \operatorname{Ker} Q.$$

By the separable case (i), it follows that $Q|_J$ does not satisfy the $\mathcal{AN}^*$ property. Consequently, Q does not satisfy $\mathcal{AN}^*$ either.

Finally, if $Q(H)$ is not countable, let $H_1 \subset Q(H)$ be an infinite countable subspace, and $Q_1 : H \to H$ be an orthogonal projection on H_1. Furthermore, let N_1 be an infinite countable subset of $Q(H)^\perp$ ($= \operatorname{Ker} Q$), and consider

$$H_2 = H_1 \oplus N_1, \tag{3.17}$$

which is a separable Hilbert space. As $H_1 = Q_1(H) \subset H_2$, by the claim above it follows that $Q_1|_{H_2} \in \mathcal{L}(H_2)$ is an orthogonal on H_2 satisfying

$$Q_1|_{H_2}(H_2) = Q_1(H) = H_1 \quad \text{and} \quad \operatorname{Ker} Q_1|_{H_2} = H_2 \cap \operatorname{Ker} Q_1. \tag{3.18}$$

Since $H_1 = Q_1(H) \subset Q(H)$ then $\operatorname{Ker} Q = Q(H)^\perp \subset Q_1(H)^\perp = \operatorname{Ker} Q_1$, we concluded by (3.17) and (3.18) that $N_1 \subset \operatorname{Ker} Q_1|_{H_2}$. Conforming with the separable case (i), it follows that $Q_1|_{H_2} \in \mathcal{L}(H_2)$ is an orthogonal projection on H_2, which does not satisfy the $\mathcal{AN}^*$ property. Consequently, since for all $x \in H_2$, $x = x_1 + x_2$, with $x_1 \in H_1 = Q_1(H) \supseteq Q(H)$, $x_2 \in N_1 \subset \operatorname{Ker} Q \subseteq \operatorname{Ker} Q_1$, thus

$$\|Qx\| = \|Q_1 x\| = \|x_1\|,$$

neither Q satisfies the $\mathcal{AN}^*$ property, and the proof is complete. $\square$

Another important characterization of $\mathcal{AN}^*$ operators is given below, but one observes first that, if $P \in \mathcal{L}(H)$ is a positive operator, then the inequality (2.6), with $y = Px$ gives

$$\|Px\|^4 \leq \langle Px, x\rangle\langle P^2x, Px\rangle \leq \langle Px, x\rangle\|P^2x\|\,\|Px\| \leq \langle Px, x\rangle\,\|P\|\,\|Px\|^2.$$

Therefore, it follows that

$$\|Px\|^2 \leq \langle Px, x\rangle\,\|P\|. \tag{3.19}$$

Now, we have the following

Lemma 3.11. *Let $K \in \mathcal{L}(H)$ be a positive compact operator and η a positive real number, such that $\eta > \|K\|/2$. Then, the operator*

$$W := \eta\, I - K,$$

satisfies the $\mathcal{AN}^*$ property.

Proof. For any $x \in S$, we have

$$\|Wx\|^2 = \eta^2 - \langle\big(2\,\eta\, K - K^2\big)x, x\rangle.$$

The condition $2\,\eta > \|K\|$ and the inequality (3.19) imply that

$$T := 2\,\eta\, K - K^2$$

is a positive compact operator, in fact for each $x \in H$

$$\langle Tx, x\rangle = 2\eta\langle Kx, x\rangle - \|Kx\|^2 \geq 0,$$

where we have used that

$$\|Kx\|^2 \leq \langle Kx, x\rangle\|K\| \leq \langle Kx, x\rangle 2\eta.$$

Now, let P_T be the positive square root of T, thus P_T is also a compact positive operator and

$$\|Wx\|^2 = \eta^2 - \|P_T x\|^2.$$

Consequently, if M is a closed subspace of H, then there exists $x_0 \in S \cap M$, such that

$$[W|_M] = \sqrt{\eta^2 - \|P_T|_M\|^2} = \sqrt{\eta^2 - \|P_T x_0\|^2} = \|Wx_0\|.$$

□

Proposition 3.12. *Let $P \in \mathcal{L}(H)$ be an $\mathcal{AN}$ orthogonal projection and $\eta > 1/2$. Then, the operator*

$$T := \eta\, I - P$$

satisfies the property $\mathcal{AN}^*$.

Proof. Since $P^2 = P$, we have for any $x \in S$

$$\|Tx\|^2 = \eta^2 + (1 - 2\,\eta)\|Px\|^2.$$

Conjointly, as P satisfies $\mathcal{AN}$ and $(1 - 2\,\eta) < 0$, then for any (closed) subspace M of H, there exist $x_0 \in S \cap M$, such that

$$\begin{aligned}{[T|_M]}^2 &= \eta^2 + (1 - 2\,\eta)\,\|P|_M\|^2 \\ &= \eta^2 + (1 - 2\,\eta)\,\|Px_0\|^2 = \|Tx_0\|^2.\end{aligned}$$

□

Proposition 3.13. *Let $P_1, P_2 \in \mathcal{L}(H)$ be $\mathcal{AN}^*$ orthogonal projections. Then $P_1 \pm P_2$, P_1P_2 and P_2P_1 satisfy the $\mathcal{AN}^*$ property.*

Proof. In fact, the proof follows with the following remark. If P is an orthogonal projection, which satisfies the $\mathcal{AN}^*$ property, then P or $I - P$ has finite rank. Therefore, if P satisfies $\mathcal{AN}^*$ or P has finite rank, or we could write $P = I - K$, where K is a projection with finite rank. Then, we conclude the proof using Remark 1.4 and Lemma 3.8. □

Acknowledgements

The authors were partially supported by Pronex-FAPERJ through the grant E-26/ 110.560/2010 entitled *"Nonlinear Partial Differential Equations"*.

References

[1] Carvajal, X., Neves, W., *Operators that achieve the norm*, Integral Equations and Operator Theory, **72** (2012), 179–195.

[2] Chevreau, B., Personal communications.

[3] Chalendar, I., Techniques d'algbres duales et sous-espaces invariants. (French) [Dual algebra techniques and invariant subspaces] With a preface in Romanian by Bernard Chevreau. Monografii Matematice (Timisoara) [Mathematical Monographs (Timisoara)], 55. Universitatea de Vest din Timisoara, Facultatea de Matematica, Timisoara, 1995. iv+94 pp.

[4] Fabian, M., Habala, P., Hjek, P., Santaluca, V.M., Pelant, J., Zizler, V., Functional Analysis and Infinite-Dimensional Geometry, CMS Books in Mathematics, Springer, New York, 2001.

[5] Halmos P.R., Introduction to Hilbert space and the theory of spectral multiplicity, Chelsea, New York, (1951).

[6] Rudin, W., Functional Analysis, McGraw-Hill, Inc., second edition, 1991.